Runge-Kutta Methods

Maharaja Agrasen University, Baddi (HP)

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2nd Order Runge-Kutta

RK2

Typical value of $\alpha = 1$, Known as RK2

Equivalent to Heun's method with a single corrector

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

Local error is $O(h^3)$ and global error is $O(h^2)$

Higher-Order Runge-Kutta

Higher order Runge-Kutta methods are available.

Derived similar to second-order Runge-Kutta.

Higher order methods are more accurate but require more calculations.

3^{rd} Order Runge-Kutta

RK3

Known as RK3

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2},\ y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + h,\ y_i - k_1 h + 2k_2 h)$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$$

Local error is $O(h^4)$ and Global error is $O(h^3)$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_2h)$$

$$k_4 = f(x_i + h, \quad y_i + k_3h)$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Higher-Order Runge-Kutta

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h)$$

$$k_4 = f(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h)$$

$$k_5 = f(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h)$$

$$k_6 = f(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h)$$

$$y_{i+1} = y_i + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$$

Example

4th-Order Runge-Kutta Method

RK4

$$\frac{dy}{dx} = 1 + y + x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Use\ RK4\ to\ compute\ \ y(0.2)\ and\ \ y(0.4)$$

Example: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ y(0.2), y(0.4)$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{h}{2},\quad y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{h}{2},\quad y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f\left(x_i + h,\quad y_i + k_3 h\right)$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Example: RK4

See RK4 Formula

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$h = 0.2$

$f(x, y) = 1 + y + x^2$

$x_0 = 0, \quad y_0 = 0.5$

Step 1

$$k_1 = f(x_0, y_0) = (1 + y_0 + x_0{}^2) = 1.5$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + (y_0 + 0.15) + (x_0 + 0.1)^2 = 1.64$$

$$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + (y_0 + 0.164) + (x_0 + 0.1)^2 = 1.654$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + (y_0 + 0.16545) + (x_0 + 0.2)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8293$$

Example: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$h = 0.2$

$f(x, y) = 1 + y + x^2$

$x_1 = 0.2, \quad y_1 = 0.8293$

Step 2

$$k_1 = f(x_1, y_1) = 1.7893$$

$$k_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h) = 1.9182$$

$$k_3 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h) = 1.9311$$

$$k_4 = f(x_1 + h, y_1 + k_3 h) = 2.0555$$

$$y_2 = y_1 + \frac{0.2}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.2141$$

Example: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

$Use\ RK4\ to\ find\ \ y(0.2), y(0.4)$

Summary of the solution

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141